Gravitational Collapse and Cosmological Constant

S. S. Deshingkar[1]*, S. Jhingan[2]†, A. Chamorro[3]‡ and P. S. Joshi[4]§

[2]Tata Institute of Fundamental Research,

Homi Bhabha Road, Mumbai 400005, INDIA

[2]Departamento de Fisica Teorica, Universidad del Pais Vasco,

Apdo. 644, E-48080 Bilbao, Spain.

February 7, 2008

---

*shrir@relativity.tifr.res.in

†wtpsaxxj@lg.ehu.es

‡wtpchbea@lg.ehu.es

§psj@tifr.res.in

Abstract

We consider here the effects of a non-vanishing cosmological term on the final fate of a spherical inhomogeneous collapsing dust cloud. It is shown that depending on the nature of the initial data from which the collapse evolves, and for a positive value of the cosmological constant, we can have a globally regular evolution where a bounce develops within the cloud. We characterize precisely the initial data causing such a bounce in terms of the initial density and velocity profiles for the collapsing cloud. In the cases otherwise, the result of collapse is either formation of a black hole or a naked singularity resulting as the end state of collapse. We also show here that a positive cosmological term can cover a part of the singularity spectrum which is visible in the corresponding dust collapse models for the same initial data.

1 Introduction

The gravitational collapse of a dust cloud has been studied extensively in the literature [1], especially in the light of the cosmic censorship conjecture of Penrose [2]. There are no globally regular solutions in these models, and a singularity always develops in an initially collapsing configuration. These singularities could be naked, or hidden behind an event horizon, depending on the nature of the initial data functions representing the density and velocity profiles for the collapsing cloud [1], from which the collapse evolves. It is also known that the shell-focusing singularities in dust models are strong curvature singularities [3]. In this sense these are physically genuine singularities and the possibility of extension of the spacetime through the same does not arise.

In investigations such as above, one generally uses the Einstein equations with a vanishing cosmological term. Recent observations however give us an evidence that a large part of the universe is possibly dominated by an energy component with negative pressure [4, 5]. Among

various possibilities available [6], one candidate that is often considered is a cosmological constant [7], Λ, or vacuum energy density corresponding to a positive sign of Λ. Such a cosmological constant would represent a spatially uniform energy density distribution, which is time-independent, and its positive value acts as a globally repulsive force field. Also, the cold dark matter models with a substantial component supplied by the cosmological term are among the models which best fit the observational data [8]. Therefore it is of value to revive the cosmological term, as a constant or even as a time varying quantity, in the Einstein equations. With such a perspective in mind we have investigated here the gravitational collapse, and structure of the singularity in the spherically symmetric dust models with the presence of a non-vanishing cosmological constant.

Dust models with a cosmological term are also known in the literature [9]. These models can be matched with the Schwarzschild-de Sitter spacetime at the boundary of the cloud [10]. But the studies so far have been restricted to special cases, whereas we analyze here the Lemaitre-Tolman-Bondi (LTB) collapse models [11] with a cosmological constant, which provide the general solution to Einstein equations with dust as the source term. While studying these models, the weak energy condition, i.e. $T_{ij}V^iV^j \geq 0$, for all non-spacelike vectors V^i, is assumed everywhere in spacetime for the matter, which would be a physically reasonable condition to assume for the case of a collapsing cloud.

In the next Section 2 we introduce the basic model and analyze the Einstein equations to check whether we can have globally regular solutions, as shown to exist in the case of homogeneous density dust solutions with a positive cosmological constant [12]. We also derive

the general solution to the Einstein equations in the case corresponding to the marginally bound case in the LTB models (i.e. the case when energy function $f = 0$), and discuss the condition for avoiding shell-crossings. In Section 3 we investigate the structure of the singularity by studying (O)outgoing (Ra)dial (N)ull (Ge)odesics (ORANGEs) near the same. The bearing of the initial conditions - in the presence of the cosmological term - on whether the final state results in a black hole or a naked singularity is then analyzed. Section 4 provides the conclusions.

2   Dust collapse with Λ term

In this section we first study the basic set of Einstein equations and the regularity conditions for collapse. Then we will look for the possibility of regular solutions when there is a non-vanishing cosmological term present, wherein an initially collapsing cloud rebounces at some later epoch so that singularity does not form. This corresponds to the occurrence of three phases, which are, collapse, reversal and subsequent dispersal. Finally, we give here a general solution with non-zero Λ term for the marginally bound case.

2.1   Basic equations and regularity

The model for a self-gravitating, spherically symmetric, inhomogeneous dust cloud with cosmological constant is given by the metric,

$$ds^2 = -dt^2 + \frac{R'^2}{1 + f(r)}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

4

where (t, r, θ, ϕ) is a comoving coordinate system. Here $f(r) > -1$, and r and R are the shell-labeling comoving coordinate, and the physical area radius respectively. In our notation the dot and the prime denote partial derivatives with respect to t and r respectively. The equation of state for matter in the interior of the cloud is that of dust (corresponding to the approximation of a fluid with negligible pressures), and the stress-energy tensor is given by,

$$T^i_j = \epsilon(r, t)\delta^i_t \delta^t_j,$$

where $\epsilon(r, t)$ is the energy density of matter. We assume the matter to satisfy the weak energy condition, which implies that $\epsilon(r, t) \geq 0$; it is equivalent to the strong energy condition as the principal pressures are zero. The Einstein equations in the presence of a cosmological constant can be written as,

$$\dot{R}^2 = \frac{F(r)}{R} + f(r) + \frac{\Lambda}{3}R^2, \tag{2}$$

$$\epsilon(t, r) = \frac{F'}{R^2 R'}. \tag{3}$$

We will be mainly considering the collapse situation (i.e. $\dot{R} < 0$). The dust cloud is characterized by two free functions in general, representing the total mass $(F(r))$, and the energy $(f(r))$, inside the shell labeled by comoving coordinate r. The cosmological term Λ can in principle be of either sign, however the recent observations as indicated above seem to favour the positive sign. The above equation for $\dot{R}$ can be in principle integrated, and after integration one gets a constant of integration. One can fix the constant of integration by using the scaling freedom. We fix this by setting $R = r$ on the initial hypersurface $(t = 0)$. The two free functions $F(r)$ and $f(r)$ can be fixed by prescribing the initial density and velocity profiles through (3) and (2) respectively. We assume these free functions to have

the form

$$F(r) = F_0 r^3 + F_n r^{3+n} + higher\ order\ terms,$$
$$f(r) = f_0 r^2 + f_{n_1} r^{2+n_1} + higher\ order\ terms,$$

(4)

where the choice of the first non-vanishing term is made in order to have a regular initial data

on the initial surface $t = 0$ from which the collapse evolves [1]. From (3) it is clear that we can

have both shell-crossing $(R'(t_{sc}(r), r) = 0)$, and shell-focusing singularities $(R(t_{sf}(r), r) = 0)$

in these spacetimes, depending on the dynamics of the shells given by (2). In this paper we

will only consider cases where there are no shell-crossings, because these are generally not

considered to be genuine singularities, and as our main interest is in studying the nature of

physical singularity corresponding to $R = 0$ where the matter shells shrink to zero radius.

This puts some restrictions on our initial data which we will discuss later.

The metric exterior to the collapsing cloud has the Schwarzschild-de Sitter form,

$$ds^2 = -g dt^2 + g^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(5)

where $g = 1 - 2M/r - \Lambda r^2/3$. Matching the solutions (1) and (5) at the boundary r_b of the

collapsing dust cloud one obtains $2M = F(r_b)$ [13, 12]. It is known that the boundary can be

made to bounce from an initially collapsing phase by choosing appropriate initial mass and

cosmological term, as we see bellow. This behaviour can also be understood by analyzing

the curve of allowed motion (2) (for details see, [13], [12]).

2.2 Regular solutions and rebounce

First we look for the possibility of having globally regular solutions due to the presence of a cosmological constant. The equation (2), governing the dynamics of collapsing shells, can be conveniently written as

$$\dot{R}^2 = \frac{V(R,r)}{3R}. \tag{6}$$

Here $V(R,r)$, defined as

$$V(R,r) = 3F(r) + 3f(r)R + \Lambda R^3, \tag{7}$$

is an analogue of the Newtonian effective potential governing motion of the shells. The allowed region of motion corresponds to $V \geq 0$ as $\dot{R}^2$ has to be greater than zero.

If we start from an initially collapsing state, we will have a rebounce if we get $\dot{R} = 0$ for a given shell before that shell becomes singular. This can happen only if the equation

$$3R\dot{R}^2 = V(R,r) = 0$$

has two real positive roots. In what follows it is convenient to define a quantity $\zeta(t,r) = R/r$. The equation (7) can be rewritten as

$$V = 3F + 3f\zeta r + \Lambda \zeta^3 r^3. \tag{8}$$

which is a cubic equation with three roots in general. ¿From the theory of cubic equations, if all the three roots of the equation above are real then at least one of them has to be positive and at least one negative. Note that $V(R = 0, r > 0) = 3F > 0$. Hence, any regular region between $R = 0$ and the first zero of eqn. (8), i.e., $\zeta_1 > 0$, always becomes singular during

7

collapse and so we need two real positive roots for the above equation (8) for a possibility of rebounce. The region between the two real positive roots is forbidden region. So if we start on the right side, the collapsing shells bounce back and then we have continuous expansion. Since one of the real root has to be negative and the region between two real positive roots is forbidden, it is not possible to have oscillating solutions in these spacetimes.

In the case of $\Lambda = 0$ it is well-known that we cannot have a rebounce and the collapse necessarily results in a singularity. The cubic then reduces to a linear equation and the solution is given as, $R = R_{max}(r) = -F/f$. So only in the case $f < 0$ we can have $\dot{R} = 0$ for positive R. This corresponds to the maximum possible physical radius for a given shell i.e. even if we start with initial expansion a given shell will reach the maximum radius $R_{max}(r)$ and then it will recollapse.

In the case when $\Lambda < 0$, one and only one root is always positive. The other two roots are negative if $9F^2 < -(4f^3/\Lambda)$, or else they are complex conjugates. Therefore any initial configuration becomes singular in this case. The real positive root in this case gives an upper bound on the radius $R = R_{max}(r)$ of a shell labeled r. This upper bound occurs as the negative (attractive) contribution from the Λ term keeps on increasing with an increasing R, while the contribution from gravitational attraction keeps on decreasing and so at some point for any value f the attraction due to Λ starts dominating. Hence, even if we have an initially expanding configuration, finally we always must have collapse in this case.

For $\Lambda > 0$ and $f(r) \geq 0$ one can easily see that we can never have $\dot{\zeta}^2 = \dot{R}^2/r^2 = 0$, so

singularity always forms if we start from an initial collapse.

For the case $\Lambda > 0$ and $f(r) < 0$, one root of the cubic above is always negative. If

$$F^2 > -\frac{4f^3}{9\Lambda},$$ (9)

then the other two roots are complex conjugates. So the singularity always forms in such a case in initially collapsing configuration. On the other hand, if the initial data is such that

$$F^2 < -\frac{4f^3}{9\Lambda},$$

then the other two roots are real positive. Let us denote them by ζ_1 and ζ_2 with $\zeta_1 < \zeta_2$. The region between the two roots is forbidden. The entire space of allowed dynamics is given by the two disjoint regions $[0, \zeta_1]$ and $[\zeta_2, \infty]$. If the initial scale factor ζ_0 lies in the first section, i.e. if $\zeta_0 < \zeta_1$, then we always have the singularity as the end point of collapse. Here $r\zeta_1$ represents the upper bound for the physical radius of a shell in this region. If ζ_0 lies in the second section, i.e. $\zeta_0 > \zeta_2$, then we will have a rebounce from the initial collapsing configuration. After the rebounce the physical radius of the shell keeps on increasing forever. There is no upper limit for the maximum value of ζ in this region and $r\zeta_2$ gives the lower bound for the physical radius of a shell, i.e. the shell rebounces at $R = r\zeta_2$.

¿From the above discussion we can see that we can have rebounce only in the case when $\Lambda > 0$ and $f < 0$, and when the following two conditions are satisfied,

$$F^2 < -\frac{4f^3}{9\Lambda},$$ (10)

Λ	$f > 0$	$f = 0$	$f < 0$
$\Lambda < 0$	Closed solution, No rebounce	Closed solution, No rebounce	Closed solution, No rebounce
$\Lambda = 0$	Open (hyperbolic) solution, No rebounce	Marginal (parabolic) solution, No rebounce	Closed (elliptic) solution, No rebounce
$\Lambda > 0$	Open solution, No rebounce	Open solution, No rebounce	Various cases occur as in table 2.

Table 1: The various types of dust solutions with Λ

$F^2 > -\frac{4f^3}{9\Lambda}$	$-$	Open solution, No rebounce
$F^2 < -\frac{4f^3}{9\Lambda}$	$\zeta_0 < \zeta_1 \Rightarrow$ $1 < 2\left(-\frac{f}{r^2\Lambda}\right)\cos\left[\frac{4\pi}{3} + \frac{1}{3}\arccos\sqrt{-\frac{9F^2\Lambda}{4f^3}}\right]$	Closed solution, No rebounce
$F^2 < -\frac{4f^3}{9\Lambda}$	$\zeta_0 > \zeta_2 \Rightarrow$ $1 > 2\left(-\frac{f}{r^2\Lambda}\right)\cos\left[\frac{1}{3}\arccos\sqrt{-\frac{9F^2\Lambda}{4f^3}}\right]$	Open Solution, Rebounce

Table 2: The various types of dust solutions with $\Lambda > 0$ and $f < 0$

and if $\zeta_0 > \zeta_2$. With our scaling, this later condition can be written as,

$$1 > 2\left(-\frac{f}{r^2\Lambda}\right)\cos\left[\frac{1}{3}\arccos\sqrt{-\frac{9F^2\Lambda}{4f^3}}\right]. \tag{11}$$

The tables 1 and 2 give a systematic classification of the various possibilities we discussed above.

The physical quantities like central density, $\rho_c = F_0/\xi^3(t)$, and curvature scalars

$$R^{ijkl}R_{ijkl} = 12\left(\frac{\dot{\zeta}^4 + \zeta^2\ddot{\zeta}^2}{\zeta^4}\right), R^{ij}R_{ij} = 12\left(\frac{\dot{\zeta}^4 + \zeta\dot{\zeta}^2\ddot{\zeta} + \zeta^2\ddot{\zeta}^2}{\zeta^4}\right), \mathcal{R} = 6\left(\frac{\dot{\zeta}^2 + \xi\ddot{\zeta}}{\zeta^2}\right)$$

stay finite as $\zeta(t,r) > 0$ for the regular models.

There are several features here which are worth noting and have interesting physical significance as far as the dynamics of collapse is concerned, and which illustrate the effects a

non-vanishing cosmological term may have towards determining the final fate of a collapsing cloud of matter. Firstly, with a negative value of the cosmological term all the solutions become closed and singularity always forms in a future even if we start with initial expansion. This is to be expected because such a value will only contribute in a positive manner to the overall gravitational attraction of matter and just acts as a constant positive energy field helping the collapse. Next, as illustrated by the first line of Table 2, there is a range of initial data where the collapse necessarily ends in a singularity despite however large positive value of the cosmological term. This is contrary to the belief sometimes expressed that a positive cosmological constant can always cause a bounce provided it is sufficiently large in magnitude. Finally, it is clear from above that there can be a rebounce only if the initial density is sufficiently low for a given positive value of Λ. This is so because the cosmological term becomes dominant with increasing distances, and gravity dominates at higher densities. Thus as the cloud is more disperse (lesser densities but bigger size), more is the effect of the cosmological term.

2.3   The $f = 0$ case

As shown in the previous consideration, while a bounce and regular solution occurs for a specific range of the initial data, for a majority of the regular initial data space the collapse results into a spacetime singularity where densities and curvatures blow up. While for $\Lambda = 0$ case we know in detail the structure of this singularity, and when it will be naked or covered. We would like to understand here the effects of a non-zero Λ towards the structure of the

singularity forming in such a collapse.

To study these features for the dust collapse models, we analyze now explicitly the case $f(r) = 0$. While we have chosen $f = 0$ for simplicity and clarity of the considerations, one expects quite similar behaviour in other cases also. The equation (2) can now be written as,

$$t - t_c(r) = \pm \int \left(\frac{F(r)}{R} + \frac{\Lambda}{3}R^2\right)^{-1/2} dR, \tag{12}$$

where $t_c(r)$ is an integration function which it represents the time at which a given shell, r, becomes singular, i.e. $R(t_{sf}(r), r) = 0$. The positive or negative sign respectively corresponds to the expanding and collapsing branches of the solution. In what follows, we consider only the negative sign because we are considering clouds which are collapsing, with $\dot{R} < 0$ initially. The integral (12) can be written as an infinite series in R near the centre as below,

$$t - t_c(r) = -\frac{2}{3}\frac{R^{3/2}}{\sqrt{F(r)}}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m(2m-1)!!}{2^m(2m+1)m!}\left(\frac{\Lambda R^3}{3F(r)}\right)^m\right]. \tag{13}$$

Using the scaling freedom in our solution, we set $R(t = 0, r) = r$ which determines $t_c(r)$ as

$$t_c(r) = t_{sf}(r) = \frac{2}{3}\frac{r^{3/2}}{\sqrt{F(r)}}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m(2m-1)!!}{2^m(2m+1)m!}\left(\frac{\Lambda r^3}{3F(r)}\right)^m\right]. \tag{14}$$

¿From the above expressions we get,

$$R' = \frac{F'}{3F}R + \left(1 - \frac{rF'}{3F}\right)\left(\frac{3F + \Lambda R^{3/2}}{3F + \Lambda r^{3/2}}\right)^{1/2}\left(\frac{r}{R}\right)^{1/2}. \tag{15}$$

Therefore, the condition $R' > 0$, implying no shell-crossings, can be satisfied if $F' > 0$ and $1 - rF'/3F > 0$. This means that the total mass inside a shell and the matter density are increasing and decreasing functions of r respectively, as we move away from the centre. The

12

weak energy condition, implying positivity of energy density, guarantees that mass is an increasing function of r. Also, for any realistic density distributions, it would be physically reasonable that the density is higher at the center, decreasing away from the center. Thus we shall work with decreasing density profiles. Therefore, no shell-crossings occur in the spacetime for our choice of initial data, before the occurrence of the central shell-focusing singularity.

As we will show in the next Section, the behaviour of collapsing shells near the centre would depend only on the first non-vanishing derivatives of the density and velocity profiles near the same. Therefore, the local visibility conditions are unaffected by the boundary conditions such as the initial choices of the mass function and the actual value ofΛ. On the other hand, the global behaviour of the trajectories coming out from the singularity can change due to the addition of a cosmological term.

3   The structure of the singularity

In this Section we will analyze the nature of the central singularity. In what follows we use the scheme, as developed earlier by Joshi and Dwivedi [15], which gives us a necessary and sufficient condition for the local visibility of the singularity. The main idea here is to see if we can have ORANGEs in the spacetime, meeting the singularity in their past with a well-defined real positive tangent vector in a suitable plane.

The equation for the ORANGEs in spacetime (1), for $f(r) = 0$ is

$$\frac{dt}{dr} = R'. \tag{16}$$

For convenience this can be written in the (u, R) plane as,

$$\frac{dR}{du} = \frac{R'}{\alpha r^{\alpha-1}} \left(1 - \sqrt{\frac{F}{R} + \frac{\Lambda}{3}R^2} \right), \tag{17}$$

where $u = r^\alpha$, and $\alpha \geq 1$ is a constant to be determined later. ¿From equation (17) it is clear that there are no ORANGEs from the non-central part of the singularity curve because the first term under square-root goes to $-\infty$ as t approaches $t_{sf}(r)$. This means that we cannot have ORANGEs as solutions to above equation near the non-central singularity, and hence it is only the central singularity which can be possibly visible.

We define $X = R/u$, to check if we can have a well-defined tangent for the equation (17), at $r = 0$, $R = 0$ in the limit of t approaching the singular epoch $t_s(0)$. Using the l'Hospitals rule we get,

$$X_0 = \lim_{u \to 0, R \to 0} \frac{R}{u} = \lim_{u \to 0, R \to 0} \frac{dR}{du} = \lim_{u \to 0, R \to 0} \frac{R'}{\alpha r^{\alpha-1}} \left(1 - \sqrt{\frac{F}{R} + \frac{\Lambda}{3}R^2} \right) = U(X_0, 0) \tag{18}$$

where the subscript 0 denotes the value of the quantities at $u = 0$. The constant α can be uniquely fixed by demanding that $R'/r^{\alpha-1}$ is non-zero finite [15]. In the case that we are discussing, $R'/r^{\alpha-1}$ remains finite if we choose $\alpha = 1 + 2n/3$ [15], and the above equation (18) can be written as,

$$\frac{1}{\alpha}\left[X - \frac{nF_n}{3F_0}\frac{1}{\sqrt{X}} \right]\left[1 + \sqrt{\frac{\lambda_0}{X}} \right] = 0, \tag{19}$$

where $\lambda = F/u$. If this equation has a real positive root X_0 then one will have at least one

14

null geodesic coming out of the singularity at $R = 0, u = 0$ with the root $X = X_0$ as a tangent in (u, R) plane.

When $n < 3$, we have $\alpha < 3$ and therefore $\lambda_0 = 0$, and the above equation reduces to,

$$X_0^{3/2} = -\frac{F_n}{2F_0\sqrt{1 + \Lambda F_0/3}} \tag{20}$$

which always has a real positive root. Apart from an additional Λ term, this equation is analogous to the corresponding equation obtained for the LTB models, and reduces to the same for $\Lambda = 0$. Therefore, when either the first or the second derivative of density is non-zero the singularity is always, at least locally, visible. The addition of cosmological term changes only the value of the tangent (X_0) to the ORANGES, but not the visibility property itself. Thus the corresponding dust naked singularity spectrum is stable to the addition of a positive cosmological constant.

In this case, like the earlier studies on LTB models, the smaller root will be along the apparent horizon direction and a family of geodesics will come out along this direction [16]. In this case we put $\alpha = 3$ and the first term in the equation (17) blows up and second term goes to zero such that the product is F_0.

For the value $n = 3$, which corresponds to the critical case in LTB models [17], we get $\lambda_0 = F_0$. Introducing $X = F_0 x^2$ and $\xi_\Lambda = F_3/F_0^{5/2}\sqrt{1 + \Lambda F_0/3}$ the root equation becomes,

$$2x^4 + x^3 - \xi_\Lambda x + \xi_\Lambda = 0. \tag{21}$$

This equation is similar to the corresponding case in the LTB models, with the modification in the definition of ξ. From the theory of quartic equations, this equation admits a real

positive root for $\xi_\Lambda < \xi_{crit} = -(26 + 15/\sqrt{3})/2$. Therefore, for a given central density F_0, and the inhomogeneity parameter as given by F_3, the naked dust singularity can be partly covered by a positive Λ, as we get an additional positive term in the denominator in ξ_Λ. But it is interesting to note that, however large, a finite Λ term cannot completely cover the corresponding visible part of dust models.

In a similar manner we can see that a negative Λ will open-up some covered part in the dust spectrum. (As such our calculation does not use the positivity of Λ anywhere, so they go through even for negative Λ.)

For all values $n > 3$ we have $\lambda_0 = \infty$, and we cannot have a real positive root in these cases and the final singularity is safely hidden behind the event horizon.

4   Conclusion

Studying gravitational collapse of a dust cloud with a non-zero value of the cosmological term we observed that black holes, naked singularities and even globally regular solutions can develop as the final outcome of collapse. Each of these outcomes is determined by the choice of initial parameters, given in terms of the density and velocity profiles of the cloud. Though for simplicity we restricted to $f(r) = 0$ case while analyzing the structure of the singularity, the results can be extended to the general case. We have also shown here that it is possible to cover a part of the naked singularity spectrum, in the corresponding critical branch of solutions in the LTB models, with the introduction of a positive cosmological

16

constant. An interesting conclusion that emerges is that the existence of naked singularity remains stable to the introduction of a cosmological term in Einstein equations. These results are of interest in that they allow us to understand the implications of a non-zero Λ towards the final outcome of gravitational collapse, in view of the recent observational claims about a non-vanishing cosmological constant.

References

[1] See, for example, S Jhingan and P. S. Joshi, Annals of the Israel Physical Society Vol. **13**, 357 (1997), and references therein.

[2] R. Penrose, Nuovo Cimento **1**, 252 (1969).

[3] S. S. Deshingkar, I. H. Dwivedi and P. S. Joshi, Phys. Rev. D**59**, 044018 (1999).

[4] See, for example, S. Perlmutter et al., Nature **391**, 51 (1998); Astrophys. J. **517**, 565 (1999), and references therein.

[5] I. Zehavi and A. Dekel, Nature **401**, 252 (1999).

[6] A. A. Starobinsky, astro-ph/9912054.

[7] V. Sahni and A. A. Starobinsky, to appear in Int. J. Mod. Phys. D (astro-ph/9904398) and references therein.

[8] See e.g. J. P. Ostriker and P. J. Steinhardt, Nature **377**, 600 (1995), and references therein.

[9] A. Krasiński, *Inhomogeneous cosmological models*, (Cambridge Univ. Press, 1997).

[10] F. Kotler, Ann. Phys. (Leipzig) **56**, 410 (1918).

[11] G. Lemaître, Ann. Soc. Sci.Bruxelles I, **A53**, 51(1933); R. C. Tolman, Proc. Natl. Acad. Sci. USA **20** (1934), 410; H. Bondi, Mon. Not. Astron. Soc. **107**(1947), 343.

[12] D. Markovic and S. L. Shapiro, Phys. Rev. D **61**, 084029 (2000).

[13] M. Cissoko, J. Fabris, J. Gariel, G. L. Denmat and N. O. Santos, gr-qc/9809057.

[14] K. Lake, Phys. Rev. D **62**, 027301 (2000).

[15] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D. **47**, 5357 (1993).

[16] S. S. Deshingkar, S. Jhingan and P. S. Joshi, Gen. Rel. Grav. **30** 1477 (1998).

[17] T. P. Singh and P. S. Joshi, Class. Quantum Grav. **13** 559 (1996); S. Jhingan, P. S. Joshi and T. P. Singh, Class. and Quant. Grav. **13**, 3057 (1996).